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Emendation of the Brown & Michael equation, with application to sound generation by vortex motion near a half-plane

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A reappraisal is made of the Brown & Michael (1954, 1955) equation that is widely used to model high-Reynolds-number vortex shedding in two dimensions by rectilinear vortices of time-dependent circulations. It is concluded that the equation introduces an unbalanced and unacceptable surface force that can significantly influence predicted flow characteristics. A corrected equation is derived which removes this force, and is applied to determine the sound generated at low Mach numbers when a line vortex translates around the edge of a rigid half-plane. The solution of this problem in the absence of vortex shedding (Crighton 1972) is extended by permitting shedding to occur at the edge in accordance with the unsteady Kutta condition. The shed vorticity is assumed to roll-up into a concentrated core whose motion is calculated by both the emended and original Brown & Michael equations. The two models exhibit large qualitative differences in the predicted wake flow near the edge; both predict significant reductions in the radiated sound, but the reduction is smaller by about 4 dB for the emended Brown & Michael equation.

1. Introduction

Two-dimensional theoretical approximations of high-Reynolds-number vortex shedding from an edge are frequently based on a formula proposed by Brown & Michael (1954, 1955) for estimating the lift on a delta wing. The continuous shedding of vorticity from the edge is modelled by means of a line (or 'point') vortex (or an appropriate sequence of vortices) whose position and circulation both depend on time, the instantaneous value of the circulation being determined by application of the Kutta condition at the edge. Separation from the edge is assumed to occur in the form of a thin sheet of vorticity of infinitesimal circulation, that rolls up into a concentrated core; the influence of vortex shedding is calculated from a potential flow representation of the interaction of this core of variable strength with the surface and any other flow structures. Brown & Michael observed that when the circulation $\Gamma(t)$ of the core varies with time t, the vortex sheet connecting it to the edge must be regarded as a 'branch-cut' across which the pressure increases discontinuously by $\rho_0\,\mathrm{d}\Gamma/\mathrm{d}t$, where ρ_0 is the density of the fluid. They claimed that the influence of this distributed force could be compensated for by requiring the core to translate at a velocity that departs from that of the fluid in its immediate neighbourhood, such that the resulting Joukowski lift on the core exactly balances the net force on the connecting sheet, and used this hypothesis to derive the equation of motion of the core.

This procedure has been refined by Mangler & Smith (1959), and by Smith (1959, 1968) in further studies of the steady-delta-wing problem, and has been extended to the related unsteady case by Dore (1966), Lowson (1963), Randall (1966) and Tavares & McCune (1993). It has also formed the basis of many numerical stimulations of unsteady vortex shedding (e.g. Clements 1973; Graham 1980; Cortelezzi & Leonard 1993; Cortelezzi, Leonard & Doyle 1994), and Rott (1956) has applied it to consider the interaction of a shock wave with an edge. Peters (1993) and Cortelezzi (1995) have demonstrated reasonable agreement between predictions of the Brown and Michael model and those obtained from more elaborate computational schemes. However, in an examination of the sound produced by vortices in low Mach number duct and cavity flows, Peters (1993) and Peters & Hirschberg (1993) have pointed out that, although the Brown & Michael formulation successfully eliminates the net force exerted on the fluid by the vortex core and connecting sheet, there exists a residual couple whose action is to create an additional surface force over and above that predicted by the usual potential flow theory. The couple is equivalent to a quadrupole source, whose influence on sound generation is ignored in the conventional theory of vortex sound. In flow of characteristic Mach number $M \ll 1$, and in the absence of solid boundaries, the efficiency of an aerodynamic quadrupole (i.e. the ratio of the acoustic power to an effective rate of production of unsteady kinetic energy in the source region) is proportional to M^5 (Lighthill 1952). When the same quadrupole is adjacent to a solid surface, however, the efficiency is usually much larger, being of order M^3 when the surface is acoustically compact (small compared to the acoustic wavelength (Curle 1955)) or of order M^2 when the quadrupole is near the edge of a large, nearly planar surface whose edge thickness is compact (Ffowcs Williams & Hall 1970; Crighton & Leppington 1970, 1971). This increase in efficiency occurs because the quadrupole produces an unsteady surface force which is equivalent to a much more powerful aerodynamic sound source of dipole type.

In this paper we argue that an improved representation of vortex shedding in terms of a concentrated core and connecting sheet will be obtained if the Joukowski lift force on the core is required to balance both the net pressure force on the connecting sheet and the surface force induced by the pressure and lift forces. Then the single vortex model of continuous vortex shedding does not involve a spurious, additional surface force, and the actual force exerted on the surface can be calculated in the usual way, in terms of the velocity potential of the motion induced by the core together with contributions from any other sources of motion in the flow.

The modification of the Brown & Michael equation of motion of the vortex core is derived in §2. Application is made to a generalization of a canonical aeroacoustic problem, originally discussed by Crighton (1972), of the sound generated in the absence of mean flow when a rectilinear vortex of circulation Γ_0 parallel to the edge of a semi-infinite plane passes around the edge under the influence of image vortices. If l is the distance of closest approach of the vortex to the edge, the wavelength of the sound is proportional to the time $\approx l^2/\Gamma_0$ required for the vortex to pass by the edge, and is very much larger than the vortex distance from the edge provided $\Gamma_0/l \ll c_0$, where c_0 is the speed of sound. In a first approximation the motion of the vortex can then be calculated as for incompressible flow, and Crighton (1972) determined the resulting acoustic radiation by matching the incompressible edge flow with an appropriate outgoing solution of the acoustic wave equation. The motion was assumed to be entirely inviscid, and no attempt was made to eliminate edge singularities of the velocity and pressure. In §3 the significance of the modified Brown & Michael equation is illustrated by incorporating vortex shedding into Crighton's problem by imposing the Kutta condition at the edge. The calculation is performed for the original and modified forms of the Brown & Michael equation. In either case, vortex shedding leads to a significant reduction in the radiated sound, by about 11 dB for the original equation and 7 dB for the emended equation. Although the calculated acoustic pressure signatures are very similar, there is a profound qualitative difference between the trailing-edge wake flows predicted by the two equations.

2. The modified Brown & Michael equation

2.1. *The Brown & Michael equation*

Consider two-dimensional incompressible flow in the neighbourhood of a rigid body S (figure 1). A distribution of vortices Σ produces unsteady motion relative to S that results in vortex shedding from the edge O. The net circulation around a closed contour enclosing S and all of the vorticity may be assumed to vanish, so that the fluid is at rest at infinity. For simplicity attention is confined to the case in which S is at rest, although the following argument is easily extended without change in the conclusions to situations where S is in unsteady translational motion (or where S is immersed in a pulsatile flow that might be produced by long wavelength sound).

Following Brown & Michael (1954, 1955) vortex shedding is modelled by introducing a point vortex of variable circulation $\Gamma(t)$ whose axis is at $\boldsymbol{x} = \boldsymbol{x}_\Gamma(t)$ with respect to the rectangular coordinate system $\boldsymbol{x} \equiv (x_1, x_2)$. The vortex is 'fed' continuously with additional vorticity which passes along a connecting sheet from O. The circulation of the connecting sheet is assumed to be negligible compared to Γ. In applications Γ is required to vary monotonically with time, and the vortex is usually assumed to be 'released' from the edge if and when $\mathrm{d}\Gamma/\mathrm{d}t$ changes sign, following which it moves with the flow as a 'free' vortex with Γ equal to its value at the time of release. The vortex then becomes a member of the distribution Σ, and a new vortex is released from the edge. We are here concerned only with the motion of Γ prior to release.

Let $\boldsymbol{v}(\boldsymbol{x}, t)$ denote the fluid velocity, and $\boldsymbol{V}(t) \equiv \mathrm{d}\boldsymbol{x}_\Gamma/\mathrm{d}t$ the translational velocity of Γ. The motion is assumed to be inviscid, except insofar as viscosity ultimately is responsible for the generation of vorticity at O at a rate determined by the Kutta condition. The vortex Γ and the connecting sheet do not obey the inviscid momentum equation

$$\rho_0 D\boldsymbol{v}/\mathrm{D}t + \nabla p = 0, \tag{2.1}$$

where ρ_0 is the fluid density and p denotes pressure. They can be excluded from the domain of validity of (2.1) by enclosing them within a control surface $f(\boldsymbol{x}, t) = 0$, shown by the broken curve in the figure, which moves relative to the fluid so as always to enclose Γ and the connecing sheet. All of the remaining vortices may be supposed to lie outside the control surface. It may be assumed that $f(\boldsymbol{x}, t) \gtrless 0$ according as $\boldsymbol{x}$ lies in the exterior/interior of the control surface, and a formal extension of the range of application of (2.1) to the whole fluid region can be derived by multiplying by $H(f)$, where $H(x)$ is the Heaviside unit function ($= 1$ or 0 according as $x \gtrless 0$), and rewriting the equation in the form

$$\rho_0 \frac{\mathrm{D}}{\mathrm{D}t}\{H(f)\,\boldsymbol{v})\} + \nabla\{H(f)\,p\} = \rho_0\,\boldsymbol{v}\frac{\mathrm{D}H}{\mathrm{D}t}(f) + p\nabla H(f). \tag{2.2}$$

The terms on the right-hand side of this equation represent forces distributed over the control surface, since $\nabla H = \nabla f \delta(f)$, and $\mathrm{D}H(f)/\mathrm{D}t \equiv (\boldsymbol{v} - \bar{\boldsymbol{V}}) \cdot \nabla H(f)$, where $\bar{\boldsymbol{V}}(\boldsymbol{x}, t)$ is the velocity of motion of the control surface, which satisfies

$$\partial f/\partial t + \bar{\boldsymbol{V}} \cdot \nabla f = 0.$$

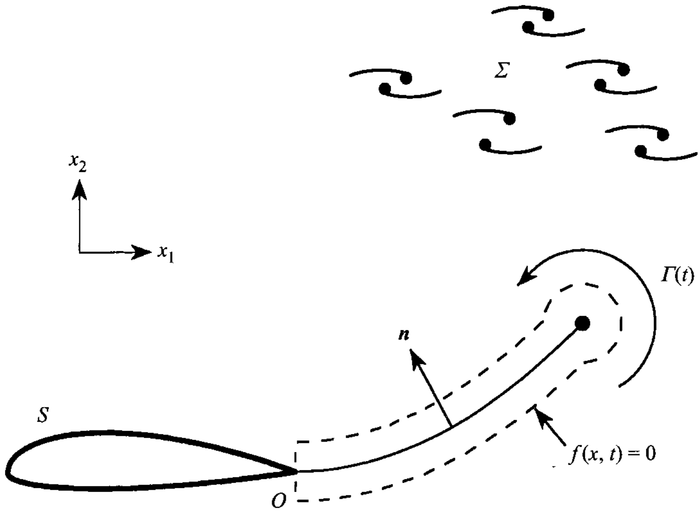

FIGURE 1. Schematic of the Brown & Michael model of vortex shedding from the edge of a surface S.

These forces can be evaluated by introducing explicit expressions for the pressure and velocity $\boldsymbol{v}$. Let Γ be represented by the concentrated vorticity distribution

$$\boldsymbol{\Omega} = \Gamma \boldsymbol{k} \delta(\boldsymbol{x} - \boldsymbol{x}_\Gamma), \tag{2.3}$$

where $\boldsymbol{k}$ is a unit vector directed out of the plane of the paper in figure 1. The velocity in $f > 0$ in the neighbourhood of Γ is then given by

$$\boldsymbol{v} \approx \boldsymbol{v}_0 + \frac{\Gamma \boldsymbol{k} \wedge (\boldsymbol{x} - \boldsymbol{x}_\Gamma)}{2\pi |\boldsymbol{x} - \boldsymbol{x}_\Gamma|^2}, \tag{2.4}$$

where $\boldsymbol{v}_0$ denotes the fluid velocity when the local (free field) velocity induced by Γ is discarded, and includes, for example, contributions induced by the vortices Σ, by images in the surface, and by any mean flow. A 'free' vortex of constant circulation at $\boldsymbol{x} = \boldsymbol{x}_\Gamma$ would translate at velocity $\boldsymbol{v}_0$. Similarly, the pressure may be calculated from Bernoulli's equation: $p/\rho_0 = -\partial\phi/\partial t - \frac{1}{2}v^2$, where in the immediate vicinity of Γ the leading-order contribution from the velocity potential ϕ is

$$\phi \approx \operatorname{Re}\{(-\mathrm{i}\Gamma/2\pi) \ln (z - z_\Gamma)\}, \quad z = x_1 + \mathrm{i}x_2, \quad z_\Gamma = x_{\Gamma 1} + \mathrm{i}x_{\Gamma 2}. \tag{2.5}$$

If the control surface is allowed to shrink down to the connecting sheet and vortex Γ, the limiting form $\boldsymbol{F}_\Gamma(\boldsymbol{x}, t)$, say, of the right-hand side of (2.2) may be found by substituting these expressions for $\boldsymbol{v}$ and p and integrating over a neighbourhood containing the control surface. By hypothesis, the velocity is continuous across the connecting sheet, but the velocity potential is discontinuous because of the time dependence of Γ in the component of ϕ shown explicitly in (2.5). An elementary calculation shows that

$$\boldsymbol{F}_\Gamma = \rho_0 \boldsymbol{\Omega} \wedge (\boldsymbol{v}_0 - \boldsymbol{V}) - \rho_0 \frac{\mathrm{d}\Gamma}{\mathrm{d}t} \boldsymbol{n} \delta(x_\perp) H(s_\Gamma - s), \tag{2.6}$$

where $x_\perp$ is distance from the connecting sheet measured in the direction of the local normal $\boldsymbol{n}$, s_Γ is the total length of the sheet (between O and Γ), and s is the distance measured along the sheet from 0. The first term on the right-hand side is concentrated at the vortex Γ, and is the reaction on the fluid to the Joukowski lift force experienced by Γ; the second term is the pressure force exerted on the fluid because of the pressure jump across the connecting sheet.

Brown & Michael (1954, 1955) choose the translational velocity $V \equiv \mathrm{d}\boldsymbol{x}_\Gamma/\mathrm{d}t$ of Γ by requiring the instantaneous net force $\int \boldsymbol{F}_\Gamma(\boldsymbol{x},t)\,\mathrm{d}x_1\,\mathrm{d}x_2$ to vanish. The integration is readily performed (with the aid of (2.3)) and yields the Brown & Michael equation

$$\frac{\mathrm{d}\boldsymbol{x}_\Gamma}{\mathrm{d}t}+\frac{\boldsymbol{x}_\Gamma}{\Gamma}\frac{\mathrm{d}\Gamma}{\mathrm{d}t}=\boldsymbol{v}_0. \tag{2.7}$$

2.2. *The emended Brown & Michael equation*

According to Brown & Michael, when $\boldsymbol{x}_\Gamma$ is determined by (2.7) the fluid motion may be calculated in terms of the velocity potential associated with the point vortices Σ and Γ (including appropriate contributions from images in S) by the usual method of potential flow theory. However, condition (2.7) does not guarantee the vanishing of the aggregate force exerted on the fluid by $\boldsymbol{F}_\Gamma(\boldsymbol{x},t)$, since this distributed system of forces generally involves an unbalanced couple which generates a reaction force $\boldsymbol{F}_\Gamma^S$ on S whose influence is not included in the potential flow equations. Thus, if it is desired to model the principal effects of vortex shedding solely in terms of the velocity potential of Γ and the free vortices Σ it is actually necessary to require that

$$\boldsymbol{F}_\Gamma^S+\int \boldsymbol{F}_\Gamma(\boldsymbol{x},t)\,\mathrm{d}x_1\,\mathrm{d}x_2=0. \tag{2.8}$$

This condition implies that the Brown & Michael equation of motion (2.7) must be modified, and to do this it is convenient to cast the momentum equation (2.2) (with the right-hand side replaced by (2.6)) in Crocco's form:

$$\frac{\partial}{\partial t}\{H(f)\,\boldsymbol{v}\}+\nabla\left[\left(\frac{p}{\rho_0}+\tfrac{1}{2}v^2\right)H(f)\right]=-\boldsymbol{\omega}_\Sigma\wedge\boldsymbol{v}H(f)+\tfrac{1}{2}v^2\nabla H(f)-\boldsymbol{v}(\boldsymbol{v}\cdot\nabla)\,H(f)+\frac{\boldsymbol{F}_\Gamma}{\rho_0},$$

where $\boldsymbol{\omega}_\Sigma=\operatorname{curl}\boldsymbol{v}$ is the vorticity field Σ. The limiting form of the right-hand side as the control surface $f=0$ shrinks down to the connecting sheet and vortex Γ is easily calculated by using (2.4), which yields

$$\frac{\partial \boldsymbol{v}}{\partial t}+\nabla\left(\frac{p}{\rho_0}+\tfrac{1}{2}v^2\right)=-\boldsymbol{\omega}\wedge\boldsymbol{v}+\frac{\boldsymbol{F}_\Gamma}{\rho_0}, \tag{2.9}$$

where $\boldsymbol{\omega}=\boldsymbol{\omega}_\Sigma+\boldsymbol{\Omega}$. In the usual way, the fluid velocity $\boldsymbol{v}$ on the right-hand side is evaluated at a point vortex by excluding the free-field, self-potential contribution of the vortex. In the particular case of the vortex Γ, $\boldsymbol{v}$ is not the same as the translational velocity V of Γ unless Γ is constant.

The momentum equation in the form (2.9), but in the absence of $\boldsymbol{F}_\Gamma/\rho_0$, was used by Howe (1989) to determine the component F_i of the net force $\boldsymbol{F}$ exerted on the fluid; the argument is unchanged by the presence of the additional term on the right-hand side, and supplies

$$F_i=-\rho_0\int \boldsymbol{\omega}\wedge\boldsymbol{v}\cdot\nabla X_i\,\mathrm{d}x_1\,\mathrm{d}x_2+\int \boldsymbol{F}_\Gamma\cdot\nabla X_i\,\mathrm{d}x_1\,\mathrm{d}x_2, \tag{2.10}$$

where $X_i(\boldsymbol{x})$ depends only on the shape of the solid S, and is the velocity potential of ideal, incompressible flow past S that has unit speed in the i-direction at large distances from S and zero circulation about S. The first integral on the right-hand side is the net force on the fluid associated with the motion produced by the vorticity (including Γ);

it is equivalent to the usual result of potential flow theory. Because the fluid is at rest at infinity, and there are no external forces, this force is actually applied at the surface S. The second term is the net force on the fluid produced by the distributed force system $\boldsymbol{F}_\Gamma$ including the reaction force $\boldsymbol{F}_\Gamma^S$ at S. If we want to represent the dynamics of the flow entirely in terms of the vortices $\boldsymbol{\omega}_\Sigma$ and Γ, it is therefore necessary that

$$\int \boldsymbol{F}_\Gamma \cdot \nabla X_i \,\mathrm{d}x_1\,\mathrm{d}x_2 = 0 \tag{2.11}$$

for all directions i. This is equivalent to (2.8), where the ith component of the surface force is

$$F^S_{\Gamma i} \equiv \int \boldsymbol{F}_\Gamma \cdot \nabla(X_i - x_i)\,\mathrm{d}x_1\,\mathrm{d}x_2.$$

Condition (2.11) determines the modified equation of motion of Γ.

Equation (2.11) can be expressed in differential form by introducing the stream function $\Psi_i(\boldsymbol{x})$ conjugate to $X_i(\boldsymbol{x})$, which satisfies the Cauchy–Riemann relations

$$\frac{\partial X_i}{\partial x_1} = \frac{\partial \Psi_i}{\partial x_2}, \quad \frac{\partial X_i}{\partial x_2} = -\frac{\partial \Psi_i}{\partial x_1},$$

where it may be assumed that $\Psi_i = 0$ on S. By using these formulae and equation (2.6) it is now straightforward to derive the emended Brown & Michael equation

$$\frac{\mathrm{d}\boldsymbol{x}_\Gamma}{\mathrm{d}t}\cdot\nabla\Psi_i + \frac{\Psi_i}{\Gamma}\frac{\mathrm{d}\Gamma}{\mathrm{d}t} = \boldsymbol{v}_0\cdot\nabla\Psi_i \quad (i = 1, 2). \tag{2.12}$$

2.3. *Production of sound at low Mach numbers*

An incompressible approximation to the equations governing an unsteady flow that generates sound can be used provided the characteristic flow Mach number is sufficiently small. In this limit the momentum equation (2.9) can be combined with the equation of continuity to yield the equation of aerodynamic sound generation in the form (Howe 1975)

$$\{\partial^2/c_0^2\partial t^2 - \nabla^2\}B = \operatorname{div}(\boldsymbol{\omega}\wedge\boldsymbol{v}) - \operatorname{div}(\boldsymbol{F}_\Gamma/\rho_0), \tag{2.13}$$

where $B = p/\rho_0 + \frac{1}{2}v^2$ is the total enthalpy, and where the acoustic pressure $p \approx \rho_0 B$ in the linearly perturbed flow at large distances from the source region. The solution of this equation can be expressed in terms of a Green's function $G(\boldsymbol{x}, \boldsymbol{y}, t-\tau)$ that has vanishing normal derivative on S (Morse & Feshbach 1953), which yields in the acoustic far field

$$p(\boldsymbol{x}, t) \approx -\int \{\rho_0\boldsymbol{\omega}\wedge\boldsymbol{v} - \boldsymbol{F}_\Gamma\}(\boldsymbol{y},\tau)\cdot\frac{\partial G}{\partial \boldsymbol{y}}(\boldsymbol{x},\boldsymbol{y},t-\tau)\,\mathrm{d}y_1\,\mathrm{d}y_2\,\mathrm{d}\tau, \tag{2.14}$$

where the integration is over the fluid and all times τ.

When S is acoustically compact (small compared to the acoustic wavelength) $G(\boldsymbol{x}, \boldsymbol{y}, t-\tau)$ may be approximated by the compact Green's function, which identifies the principal source of sound with an equivalent distribution of dipoles on S, whose net strength is the unsteady surface force. For an observer at $\boldsymbol{x}$ at large distances from S, and for entirely two-dimensional motions, it is known that (Howe 1975)

$$G(\boldsymbol{x},\boldsymbol{y},t-\tau) = \frac{\boldsymbol{x}\cdot\boldsymbol{Y}}{4\pi c_0}\frac{\partial}{\partial t}\left(\frac{H(t-\tau-r/c_0)}{(t-\tau)(c_0^2(t-\tau)^2-r^2)^{1/2}}\right) \quad (r\rightarrow\infty), \tag{2.15}$$

where $r = (x_1^2+x_2^2)^{1/2}$, and $\boldsymbol{Y} = (X_1(\boldsymbol{y}), X_2(\boldsymbol{y}))$. It follows from this and the equation of motion (2.11), that the dipole sound produced by $\boldsymbol{F}_\Gamma$ vanishes identically. Thus, the emended Brown & Michael equation removes the spurious radiation noted by Peters & Hirschberg (1993) and Peters (1993) to be produced by the quadrupole formed by the Joukowski lift on Γ and the force on the connecting sheet.

3. Radiation from vortex motion near a half-plane

3.1. *No vortex shedding*

To illustrate the implications of the emended Brown & Michael formula, consider the sound generated when a point vortex of circulation Γ_0, say, translates at infinitesimal Mach number past the edge of a rigid half-plane. This problem was first formulated by Crighton (1972) who solved it by the method of matched asymptotic expansions, according to which the vortex motion is calculated in a first approximation as for an inviscid, incompressible fluid.

In the absence of vortex shedding from the edge, the motion of Γ_0 is governed by the velocity field of a distribution of images in the surface. When $\Gamma_0 > 0$, the vortex traverses in a clockwise sense the symmetric path illustrated in figure 2(*a*), where the half-plane is taken to be the negative x_1-axis and the time origin has been adjusted to coincide with the instant at which the vortex crosses the x_1-axis, when its distance from the edge is a minimum and equal to l. Far from the edge the motion is parallel to the plane at speed $U = \Gamma_0/8\pi l$, and the points marked on the trajectory indicate the position of the vortex at different non-dimensional times Ut/l. Crighton (1972) showed that the position $(X_{01}, X_{02}) \equiv (x_{01}/l, x_{02}/l)$ of the vortex is given by

$$X_{01} = \frac{1-T^2}{(1-T^2)^{1/2}}, \quad X_{02} = \frac{-2T}{(1-T^2)^{1/2}}, \quad T = Ut/l, \tag{3.1}$$

and that the leading approximation to the acoustic pressure $p(r,\theta,t)$ at large distances r from the edge in direction θ (see figure 2) is

$$p \approx 4\rho_0 U^2 \left(\frac{l}{r}\right)^{1/2} \sin\left(\tfrac{1}{2}\theta\right) \left[\frac{T}{(1+T^2)^{5/4}}\right]_{t-r/c_0} \quad (r \to \infty), \tag{3.2}$$

where the term in square brackets is evaluated at the retarded time $t-r/c_0$. The non-dimensional pressure signature $p/\{\rho_0 U^2 (l/r)^{1/2} \sin(\frac{1}{2}\theta)\}$ is illustrated in figure 2(*b*), which indicates that significant radiation is produced only during the interval $-4 < Ut/l < 4$ when the vortex is accelerating around the edge. When r is large the total radiated acoustic energy is given by

$$E_0 = \int_{-\infty}^{\infty} \int_0^{2\pi} \{p^2(r,\theta,t)/\rho_0 c_0\} r \,\mathrm{d}\theta \,\mathrm{d}t \equiv \tfrac{32}{3}\pi\rho_0 U^2 M l^2, \tag{3.3}$$

where the Mach number $M = U/c_0 \ll 1$.

When the path $\boldsymbol{x}_0(t)$ of the vortex is known the acoustic pressure is readily obtained in the form (3.2) by invoking the general formula (2.14), with $\boldsymbol{F}_\Gamma \equiv \boldsymbol{0}$ and $\boldsymbol{\omega} \equiv \Gamma_0 \boldsymbol{k}\delta(\boldsymbol{x}-\boldsymbol{x}_0)$, and substituting the following expression for the compact Green's function for a rigid half-plane (Howe 1975)

$$G(\boldsymbol{x},\boldsymbol{y},t-\tau) = \frac{\phi^*(\boldsymbol{y})}{\pi r^{1/2}} \sin\left(\tfrac{1}{2}\theta\right) \delta(t-\tau-r/c_0) \quad (r \to \infty). \tag{3.4}$$

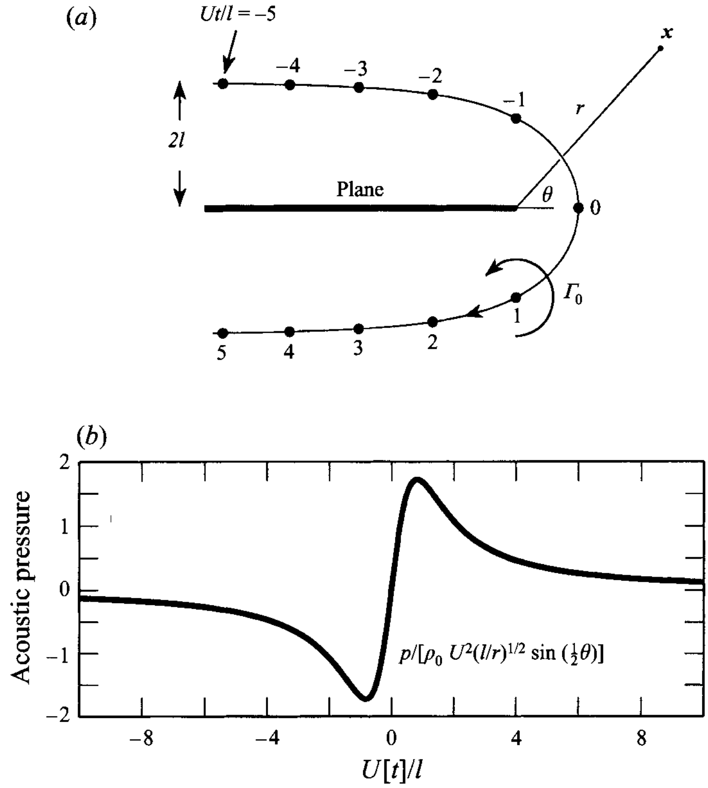

FIGURE 2. (*a*) Vortex motion in the absence of vortex shedding. l is the distance of closest approach of the vortex to the edge. (*b*) Acoustic pressure signature in the absence of shedding, as a function of the retarded time $[t] = t - r/c_0$.

The function $\phi^*(\boldsymbol{x})$ is the real part of the analytic function

$$\phi^*(\boldsymbol{x}) + \mathrm{i}\psi^*(\boldsymbol{x}) \equiv -\mathrm{i}z^{1/2}, \quad z = x_1 + \mathrm{i}x_2, \tag{3.5}$$

which will be recognized as the complex potential of ideal, incompressible flow around the edge of the half-plane along (parabolic) streamlines $\psi^*(\boldsymbol{x}) = \text{constant}$. The formula (3.4) is applicable provided the distance $|\boldsymbol{y}|$ of the aeroacoustic source from the edge is small compared to the acoustic wavelength.

By using (2.14) and (3.4), together with the Cauchy–Rieman equations, the radiation can be expressed in the alternative general form

$$p \approx \frac{\rho_0 \Gamma_0}{\pi r^{1/2}} \sin(\tfrac{1}{2}\theta)\,[\boldsymbol{v} \cdot \nabla\psi^*] \quad (r \to \infty), \tag{3.6}$$

where the term in square brackets is evaluated at the retarded position $\boldsymbol{x}_0(t - r/c_0)$ of the vortex, and $\boldsymbol{v}$ is the flow velocity at the vortex core. This result will be used below.

3.2. *Vortex shedding*

The influence of vortex shedding will be estimated by modelling the continuously shed vorticity by a concentrated core (point vortex) of circulation $\Gamma(t)$. It will be seen *a posteriori* that $\Gamma(t)$ varies monotonically with time, and therefore, that it is not necessary to postulate the shedding of more than one vortex. The instantaneous value of Γ is determined by the Kutta condition. The equations of motion of the incident and shed vortices can be derived in the usual way by mapping the z-plane ($z = x_1 + \mathrm{i}x_2$) cut along the negative real axis (the half-plane) onto the upper ζ-plane by means of the transformation $\zeta = \mathrm{i}z^{1/2}$. If z_0 and z_Γ are the respective locations of Γ_0 and Γ, the Kutta condition applied at the edge $z = 0$ yields

$$\Gamma = -\frac{\Gamma_0 |z_\Gamma|}{|z_0|}\left(\frac{(z_0)^{1/2} + {}^*(z_0)^{1/2}}{(z_\Gamma)^{1/2} + {}^*(z_\Gamma)^{1/2}}\right), \tag{3.7}$$

where the asterisk denotes the complex conjugate. This formula shows that the circulations Γ_0, Γ are always opposite in sign.

The motion of the shed vortex core is governed by (2.12). The case $i = 1$ supplies the equation for the velocity component $\mathrm{d}x_{\Gamma 2}/\mathrm{d}t$ normal to the half-plane, since the stream function of flow at unit speed parallel to the x_1-axis is $\Psi_1 \equiv x_2$. For $i = 2$, Ψ_2 must be defined by the limiting value as $a \to \infty$ of the stream function for a plate of finite length $2a$ in the x_1-direction, namely, $\Psi_2 = \lim_{a\to\infty} \mathrm{Im}\{-\mathrm{i}(z(z+2a)^{1/2}\} \approx \mathrm{Im}\{-\mathrm{i}(2az)^{1/2})\}$. However, the length a does not appear in the equation of motion, because (2.12) is homogeneous in Ψ_i. It will be observed that this limiting value of Ψ_2 is proportional to the function ψ^* in the general solution (3.6); this must be the case if no dipole radiation is to be produced by the Joukowski lift on Γ and the connecting sheet force. By setting $\mathrm{d}\Gamma/\mathrm{d}t = 0$ in (2.12), we obtain also the usual ('free' vortex) equation of motion for the incident vortex Γ_0. When the complex positions of the vortices are non-dimensionalized by the distance l of closest approach of Γ_0 to the edge in the absence of shedding (§3.1), the simultaneous equations of motion may be rearranged into the complex forms

$$\frac{\mathrm{d}Z_\Gamma^*}{\mathrm{d}T} + \frac{(|Z_\Gamma| + Z_\Gamma^*)}{\Gamma}\frac{\mathrm{d}\Gamma}{\mathrm{d}T} = \frac{\mathrm{i}(\Gamma/\Gamma_0)}{Z_\Gamma} + \frac{2\mathrm{i}(\Gamma/\Gamma_0)}{Z_\Gamma + |Z_\Gamma|} - \frac{2\mathrm{i}}{(Z_\Gamma)^{1/2}}\left(\frac{1}{(Z_\Gamma)^{1/2} - (Z_0)^{1/2}} - \frac{1}{(Z_\Gamma)^{1/2} - {}^*(Z_0)^{1/2}}\right), \tag{3.8a}$$

$$\frac{\mathrm{d}Z_0^*}{\mathrm{d}T} = \frac{\mathrm{i}}{Z_0} + \frac{2i}{Z_0 + |Z_0|} - \frac{2\mathrm{i}(\Gamma/\Gamma_0)}{(Z_0)^{1/2}}\left(\frac{1}{(Z_0)^{1/2} - (Z_\Gamma)^{1/2}} - \frac{1}{(Z_0)^{1/2} - {}^*(Z_\Gamma)^{1/2}}\right) \tag{3.8b}$$

where $Z_0 = z_0/l, Z_\Gamma = z_\Gamma/l, T = Ut/l \equiv \Gamma_0 t/8\pi l^2$.

To compare the solution of these equations with the motion of Γ_0 predicted in §3.1 in the absence of shedding, they are integrated from a large and negative value of T with the initial conditions on $Z_0 \equiv X_{01} + \mathrm{i}X_{02}$ defined by (3.1). Vortex shedding has a negligible effect on Γ_0 when T is large and negative, and the solutions of (3.8*a, b*) are insensitive to the precise initial value of $Z_\Gamma \equiv X_{\Gamma 1} + \mathrm{i}X_{\Gamma 2}$ used in the computation. We have considered two alternative sets of conditions. In the first we take $X_{\Gamma 1} = 0$, and select an arbitrarily small, but non-zero value for $X_{\Gamma 2}$. For the second approach, we observe that the final term on the right-hand side of (3.8*a*)$\to \mathrm{i}/(2(Z_\Gamma)^{1/2}|T|^{3/2})$ as $T \to -\infty$, for which equation (3.8*a*) has the similarity solution

$$Z_\Gamma = \left(\frac{17}{4\times 13}\right)^{1/3} \frac{\mathrm{e}^{-\mathrm{i}\Theta}}{|T|^{1/3}} \quad \text{where} \quad \Theta = \pi - \cos^{-1}(5/17),$$

which is then used to define Z_Γ when T is large and negative. The integrations have been performed using a standard fourth-order Runge–Kutta scheme. The procedure is stable, and the accuracy was checked by reducing the step size until successive

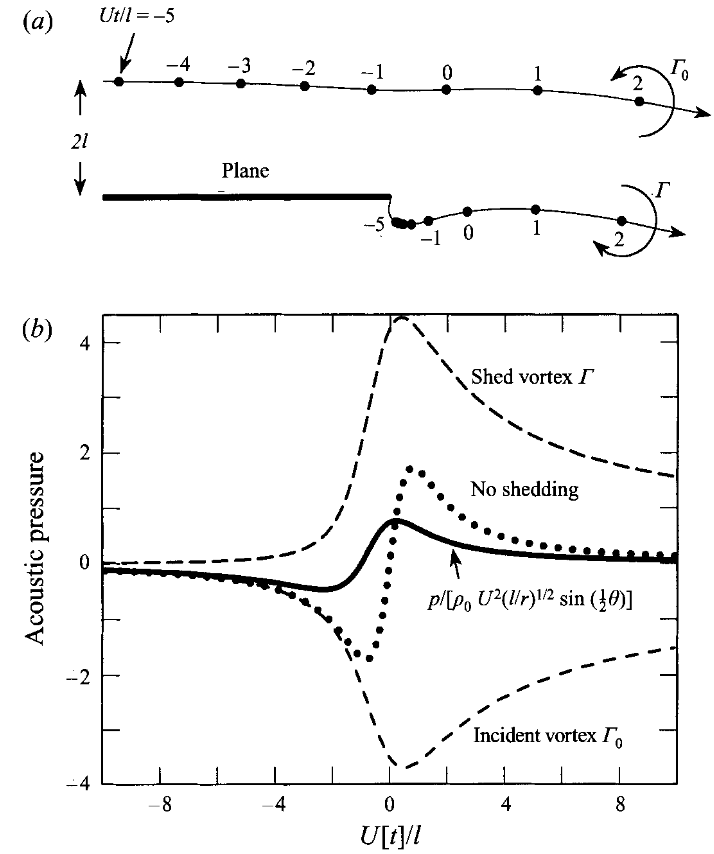

FIGURE 3. (*a*) Incident and shed vortex trajectories determined by the emended Brown & Michael equations (3.8). (*b*) ——, The acoustic pressure signature, compared with the contributions from the incident and shed vortices (broken curves) and · · ·, in the absence of vortex shedding.

predictions of Z_0 and Z_Γ differed by less than 10^{-4}. The trajectories of the incident and shed vortices are depicted in figure 3(*a*), with the vortex positions indicated at several values of T for comparison with figure 2(*a*).

Equation (3.6) for the acoustic pressure can be applied to the vortices Γ_0 and Γ separately to calculate their contributions to the sound. In each case $\boldsymbol{v}$ is the fluid velocity evaluated at the vortex core after subtracting out the self-potential contribution of the type (2.5). This velocity coincides with $\mathrm{d}\boldsymbol{x}_0/\mathrm{d}t$ for the incident vortex Γ_0, but is not equal to $\mathrm{d}\boldsymbol{x}_\Gamma/\mathrm{d}t$ for the shed vorticity, except at large times when $\Gamma(t)$ becomes constant. The solid curve in figure 3(*b*) is the net acoustic pressure signature $p/\{\rho_0\, U^2(l/r)^{1/2}\sin(\frac{1}{2}\theta)\}$; the broken curves are the separate contributions from the vortices, which are effectively equal and opposite for $T > 0$, when the amplitude of the sound becomes very much less than in the absence of shedding (shown dotted in the figure). Very little sound is generated by the shed vorticity prior to the arrival of the incident vortex at the edge, and the acoustic pressure then coincides with that in the absence of shedding.

The growth of the shed vortex circulation is depicted by the solid curve in figure 4. Shedding does not become significant until about $Ut/l = -5$, and is effectively complete by $Ut/l = 2$. At that time, the incident and shed vortex form a silent self-propelling vortex pair. Reference to figure 3(*b*) reveals that the amplitude of the sound

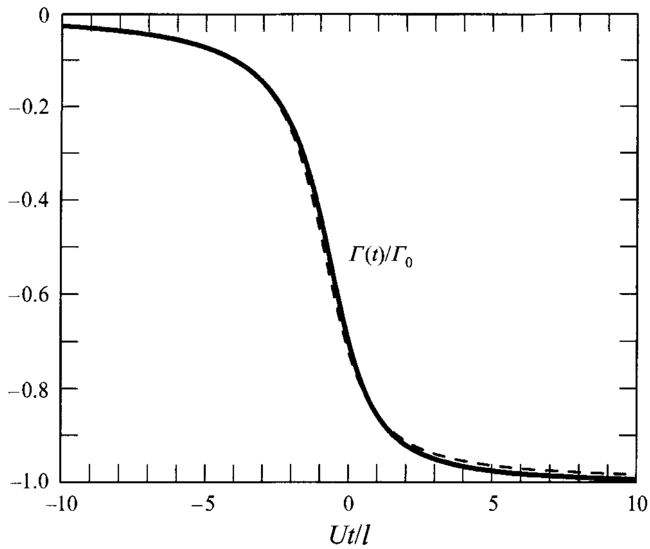

FIGURE 4. Time dependence of the circulation of the shed vortex; the broken curve is the corresponding prediction using the original Brown & Michael equation.

that each vortex would make in isolation is large; the net pressure is small, however, because both vortices are cutting across contours of constant values of ψ^* at essentially the same rate, and equation (3.6) shows that it is precisely this rate of cutting of the streamlines of an ideal edge flow that determines the amplitude of the sound.

Similar general conclusions obtain when the vortex paths are computed from the original Brown & Michael equation (2.7) (formally equivalent to omitting $|Z_\Gamma|$ from the coefficient of $\mathrm{d}\Gamma/\mathrm{d}T$ in (3.8*a*)). The variation of the shed vortex strength (see figure 4) is very similar in the two cases, but the vortex trajectories predicted by the Brown & Michael equation (figure 5*a*) are very different, and characterize a flow that is deflected sideways by the plane, of the type that might be expected if a net force were to be applied to the flow by the plane. The predicted acoustic amplitude (calculated, as before, from (3.6) with v equal to the fluid velocity at a vortex after subtraction of the singular self-potential term) is also very much smaller than for the emended equation of motion, although the pressure signatures are qualitatively the same. The integration with respect to time in (3.3) can be effected numerically to compare the radiated acoustic energy E with that in the absence of shedding. We find $E/E_0 \approx 0.18$ (≈ -7.5 dB) when the motion is governed by the emended equation (figure 3), but is much smaller, $E/E_0 \approx 0.08$ (≈ -11 dB), for the Brown & Michael formulation.

It is interesting to note that these nonlinear calculations tend to confirm the principal hypothesis of existing approximate theories of trailing-edge noise (Chandiramani 1974; Chase 1972, 1975), namely that, for a thin trailing edge immersed in an essentially uniform mean flow, sound is produced by the interaction with the edge of surface pressure fluctuations associated with nonlinear mechanisms in the boundary layer. According to linear theory, when both incident and shed vorticity are assumed to convect at the same mean free-stream velocity, the sound pressure produced at the trailing edge by the incident and shed vorticity are equal and opposite.

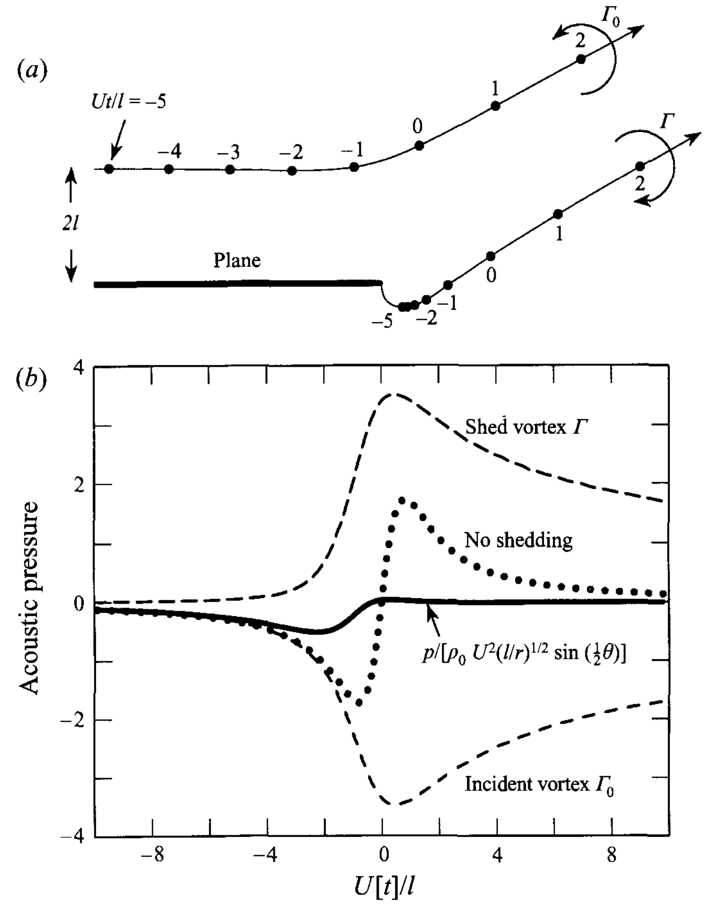

FIGURE 5. (*a*) Incident and shed vortex trajectories determined by the Brown & Michael equation. (*b*) ——, the acoustic pressure signature, compared with the contributions from the incident and shed vortices (broken curves) and · · ·, in the absence of vortex shedding.

4. Conclusion

Brown & Michael (1954, 1955) obtained an equation governing the motion of a rectilinear vortex of variable strength that has been widely applied in the numerical modelling of vortex shedding from rigid bodies in incompressible flow. It has been argued in this paper that the equation is inconsistent with the requirement that no extraneous forces be introduced by the model. An emended equation of motion has been proposed which eliminates such forces. Our application of the emended equation to the problem of sound generation by a vortex interacting with the edge of a rigid plane reveals a large qualitative difference between the predicted wake flow (i.e. the vortex trajectories) and the flow derived from the Brown & Michael equation. Both models imply that vortex shedding at the edge is responsible for a significant decrease in the radiated sound, but the reduction predicted by Brown & Michael is greater by nearly 4 dB.

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